Announcements

1) Final (time + rules)
11:30 – 2:30 Wednesday 12/21
CB 2048
(Monday classroom)

2) Final cumulative, practice final on CTools, will be a "Things you should know" worksheet, practice problems

3) office hours

12-2 Thursday

2-4 Monday

12-2 Tuesday

4) review

5-?? Monday night

with pizza

Recall volume (Disk method)

Volume obtained by revolving $y=f(x)$ from $x=a$ to $x=b$ about the x-axis is

$$\boxed{\int_a^b \pi(f(x))^2\,dx}$$

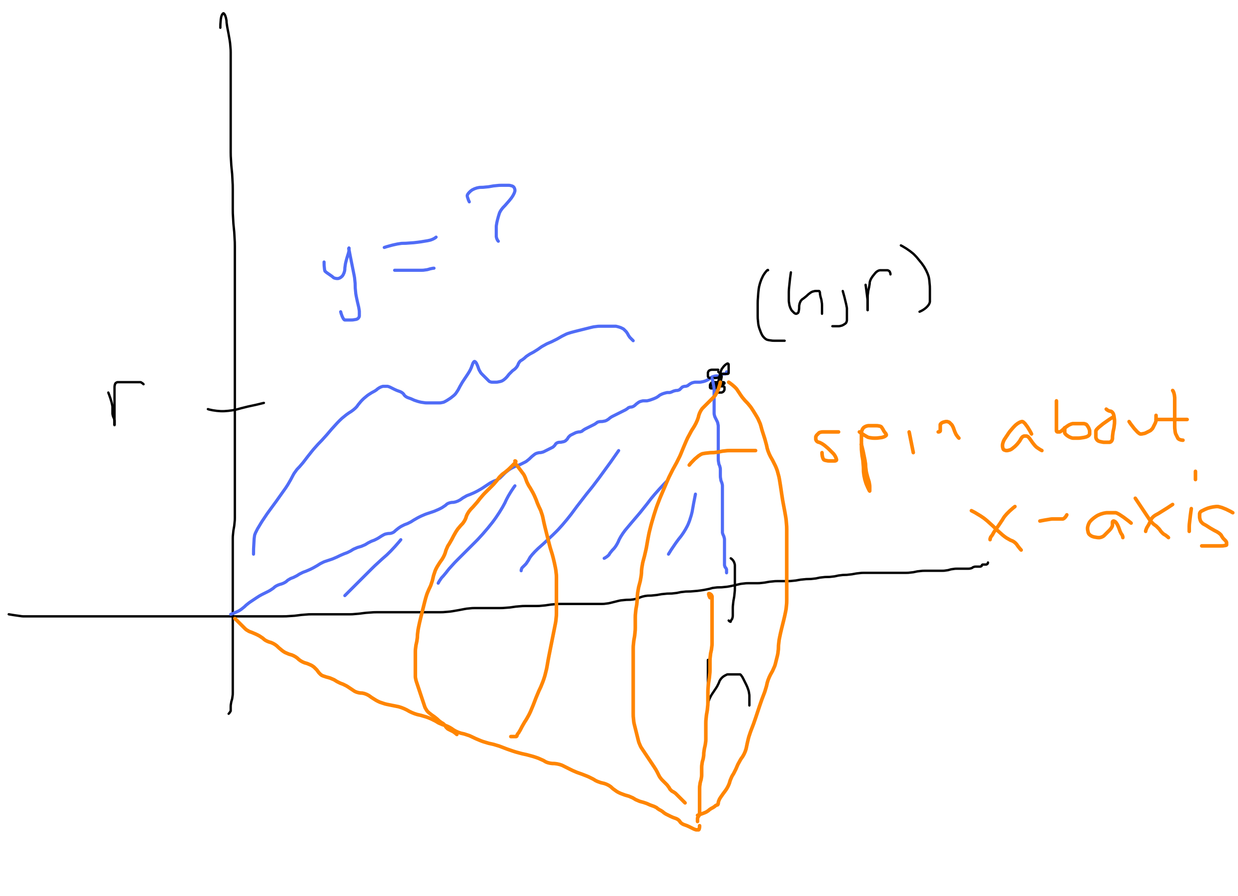
area inside disk

$(f>0)$

Example 1: Compute the volume of a right circular cone

$$V = \frac{1}{3}\pi r^2 h$$

Derive using disk method.



The slope of the line determining the blue triangle is $\frac{r}{h}$.

Since the y-intercept is zero, the equation for the line is

$$y = \frac{r}{h} x.$$

Plug this into the disk formula.

$$V = \int_a^b \pi (f(x))^2 \, dx$$

$$= \int_0^h \pi \left(\frac{r}{h}x\right)^2 dx$$

$$= \pi \int_0^h \frac{r^2}{h^2} \cdot x^2 \, dx$$

$$= \frac{\pi r^2}{h^2} \int_0^h x^2 \, dx$$

$$= \frac{\pi r^2}{h^2} \cdot \frac{x^3}{3} \Big|_0^h$$

$$= \frac{\pi r^2}{\cancel{h^2}} \cdot \frac{h^{\cancel{3}}}{3} = \boxed{\frac{1}{3}\pi r^2 h}$$

Example 2:

$$y = \frac{\sqrt{2x+1}}{4x^2+4x+13}$$

from $x = 1$ to $x = 4$

Formula for the volume of the object obtained by revolving this region about the x-axis is

$$V = \int_1^4 \pi \left(\frac{\sqrt{2x+1}}{4x^2+4x+13} \right)^2 dx$$

$$= \pi \int_1^4 \frac{2x+1}{(4x^2+4x+13)^2}\, dx$$

$$u = \boxed{4x^2+4x+13}$$

$$du = 8x+4 \quad dx$$

$$= 4(2x+1)\, dx$$

so $$\frac{du}{4} = \boxed{(2x+1)\, dx}$$

$$u(1) = 21$$

$$u(4) = 93$$

The volume becomes

$$V = \frac{\pi}{4} \int_{21}^{93} \frac{du}{u^2}$$

$$= \frac{\pi}{4} \int_{21}^{93} u^{-2}\, du$$

$$= \frac{\pi}{4} \left(-\frac{1}{u}\right) \Big|_{21}^{93}$$

$$= \frac{\pi}{4} \left(-\frac{1}{93} - \left(-\frac{1}{21}\right)\right)$$

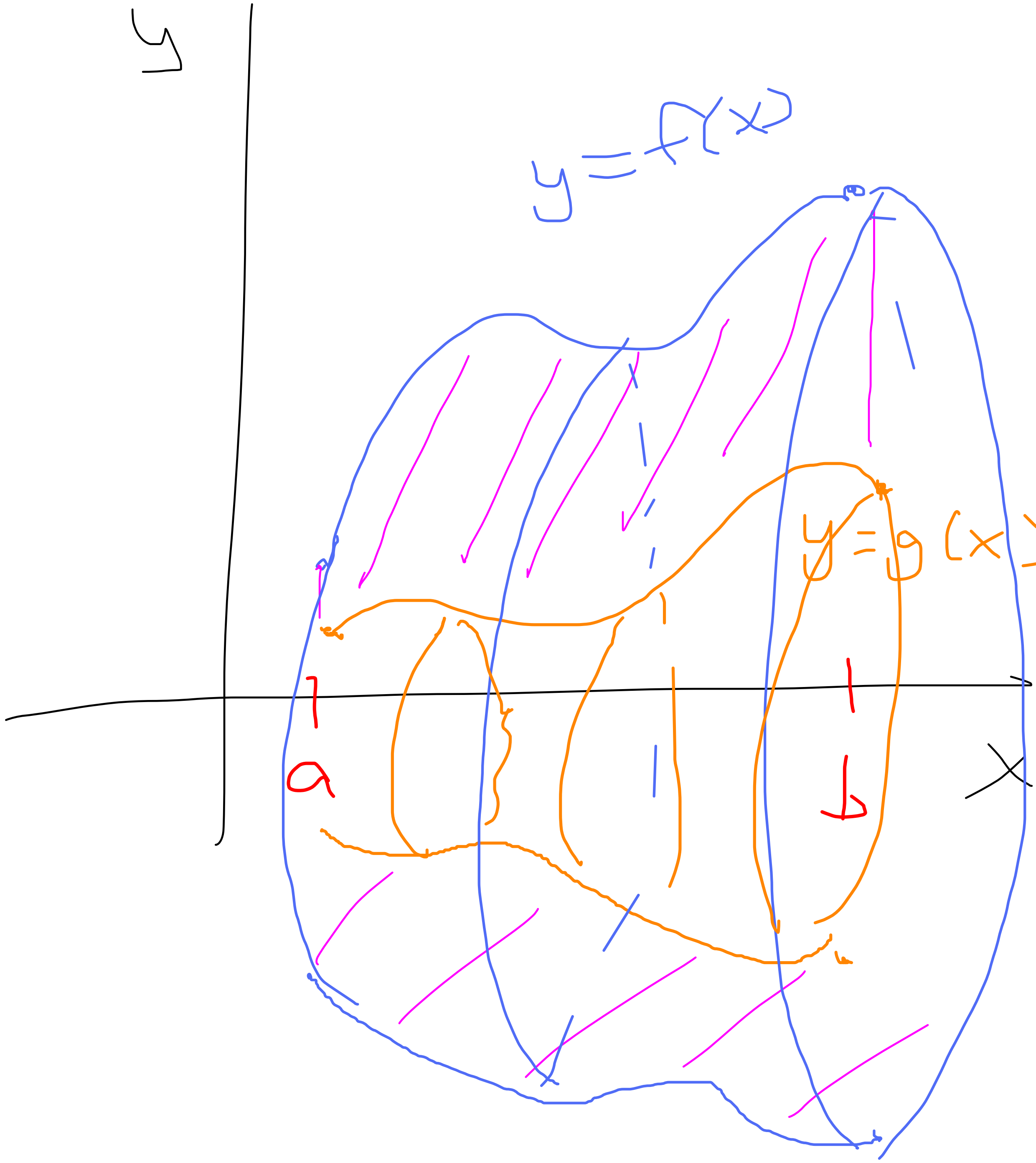
$$= \boxed{\frac{\pi}{4} \left(\frac{1}{21} - \frac{1}{93}\right)}$$

Volume generated by revolving region bounded by 2 curves

The volume of the region bounded by the curves $y=f(x)$ and $y=g(x)$ from $x=a$ to $x=b$ spun about the x-axis is

$$\boxed{V=\int_a^b \pi(f(x))^2dx-\int_a^b \pi(g(x))^2dx}$$

if $f>g$ on $[a,b]$



y
y = f(x)
y = g(x)
a
b
x

Note: If $g > f$ on $[a, b]$, you switch their positions in the volume formula.

If the curves intersect, figure out all intersection points, use them to break up the volume into multiple integrals.

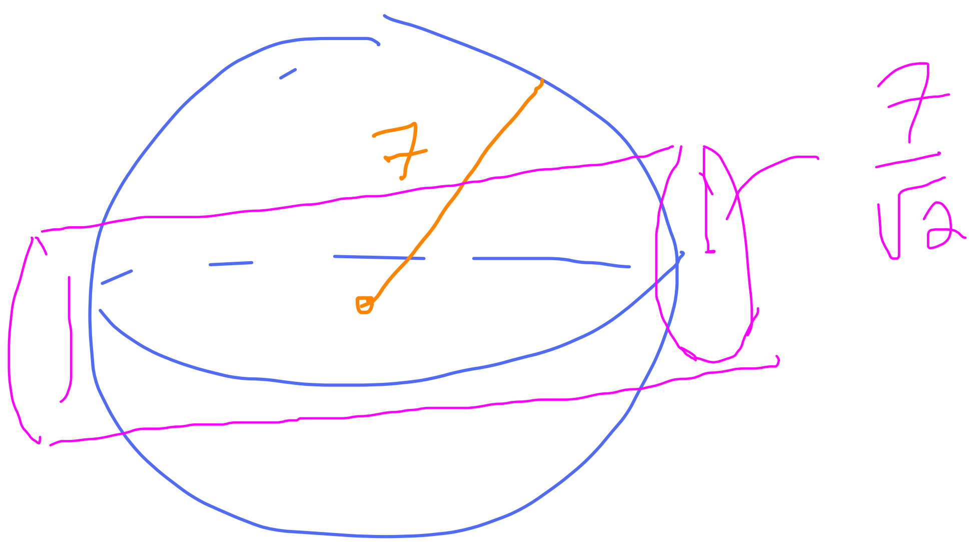
Warning: NOT same as

$$\cancel{\pi \int_a^b (f(x) - g(x))^2 \, dx}$$

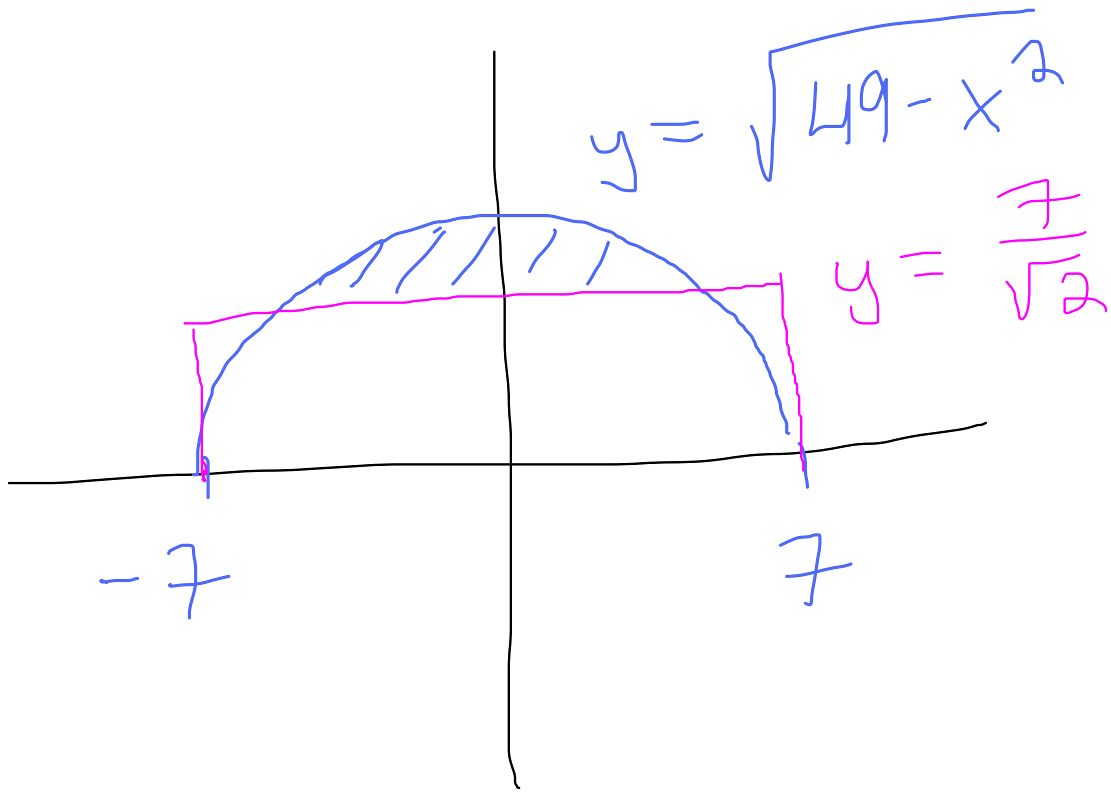
Square twice since you have two curves.

Example 3: Sphere of radius 7 cm hollowed out by cylindrical drill bit of radius

$\frac{7}{\sqrt{2}}$.



2-D picture that we will revolve:



find where the curves intersect

$\sqrt{49 - x^2} = \frac{7}{\sqrt{2}}$ square both sides

$49 - x^2 = \frac{49}{2}$, so $\boxed{x = \pm \frac{7}{\sqrt{2}}}$

$$V = \underline{\int_{-\frac{7}{\sqrt{2}}}^{\frac{7}{\sqrt{2}}} \pi \left(\sqrt{49-x^2}\right)^2 dx}$$

$$- \underline{\int_{-\frac{7}{\sqrt{2}}}^{\frac{7}{\sqrt{2}}} \pi \left(\frac{7}{\sqrt{2}}\right)^2 dx}$$

$$= \pi \int_{-\frac{7}{\sqrt{2}}}^{\frac{7}{\sqrt{2}}} (49 - x^2)\, dx = \boxed{\pi \int 49} - \int x^2$$

$$\boxed{- \pi \frac{49}{2}} \int_{-7/\sqrt{2}}^{7/\sqrt{2}} 1 \, dx$$

$$= \pi \frac{49}{2} \int_{-\frac{7}{\sqrt{2}}}^{7/\sqrt{2}} 1 \, dx$$

$$- \pi \int_{-\frac{7}{\sqrt{2}}}^{7/\sqrt{2}} x^2 \, dx$$

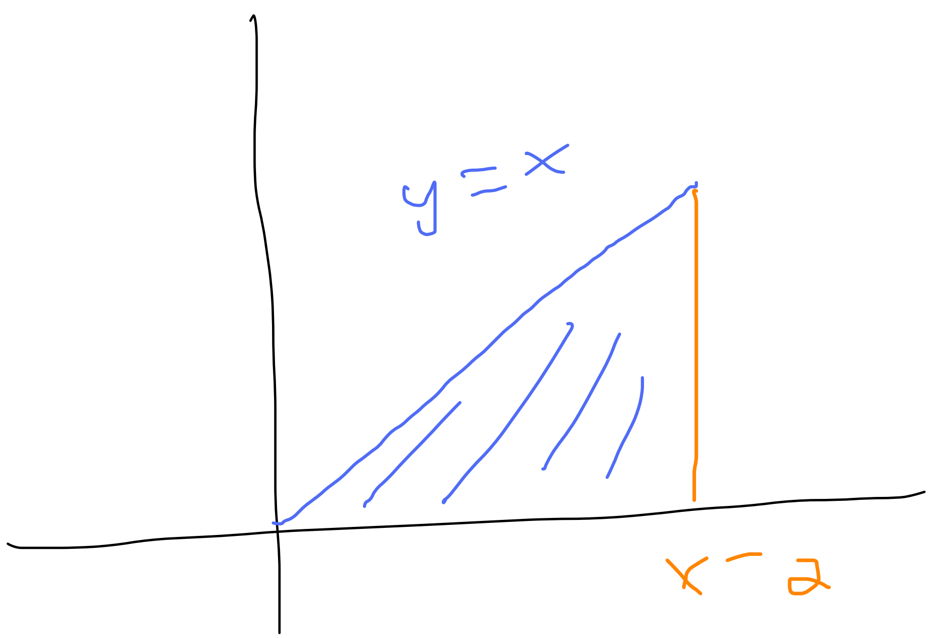
$$= \frac{49\pi}{2} \left(x \Big|_{-\frac{7}{\sqrt{2}}}^{\frac{7}{\sqrt{2}}} \right) - \pi \left(\frac{x^3}{3} \Big|_{-\frac{7}{\sqrt{2}}}^{\frac{7}{\sqrt{2}}} \right)$$

$$\boxed{= \pi \left(\frac{343}{\sqrt{2}} - \frac{343}{3\sqrt{2}} \right)}$$

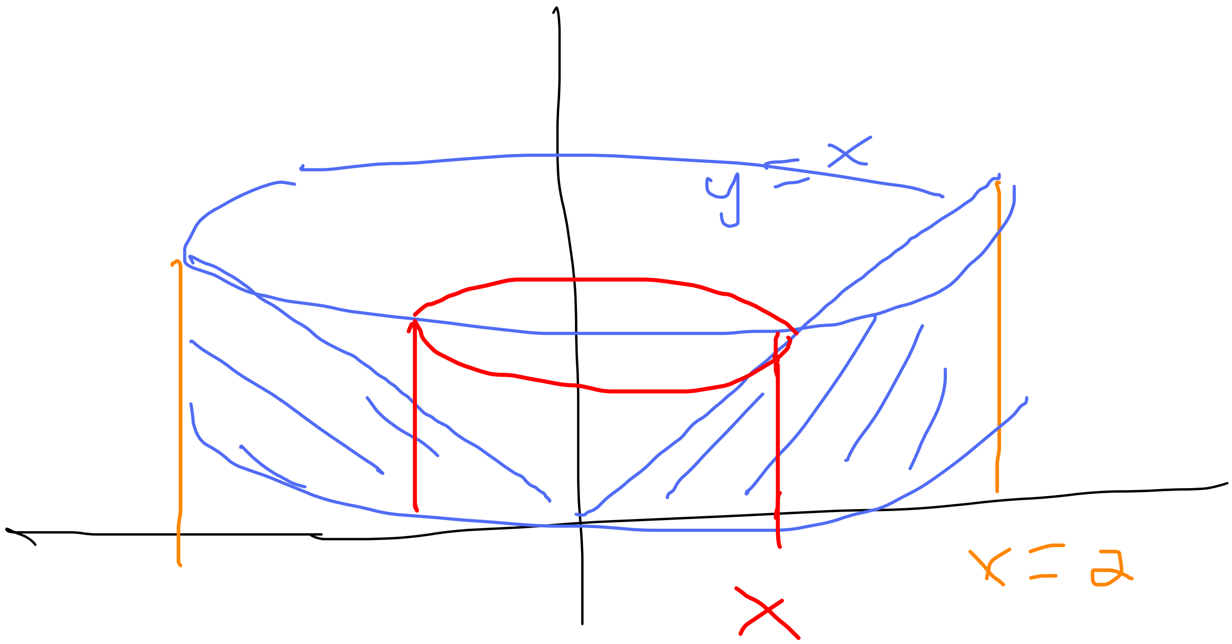
Revolving About the y-axis

$y = x$, x-axis, $x = 2$

is our region.



spin about the y axis!



a cylinder with a cone hollowed out.

look at a particular x value. Get a circle of height $f(x)$ & radius x. Area of strip formed by revolving is $2\pi x \cdot f(x)$

$$= 2\pi x^2.$$

The volume is then the integral from $x=0$ to $x=2$; $V = \int_0^2 2\pi x^2 \, dx$

Formula (cylindrical shells)

$$V = \int_a^b \underbrace{2\pi x}_{\text{circumference}} \, \underbrace{f(x)}_{\text{height}} \, dx$$

for the region bounded by $y = f(x)$, the x-axis, $x = a$, $x = b$ spun about the y axis

Example 4:

Finish

$y = x$ from

$x = 0$ to $x = 2$

$$V = 2\pi \int_0^2 x^2 \, dx$$

$$= 2\pi \left(\frac{x^3}{3} \Big|_0^2 \right)$$

$$= \boxed{\frac{16\pi}{3}}$$

2 curves

$f > g$

$$V = 2\pi\left(\int_a^b f(x)\cdot x\,dx - \int_a^b x g(x)\,dx\right)$$

$g > f$, switch their roles

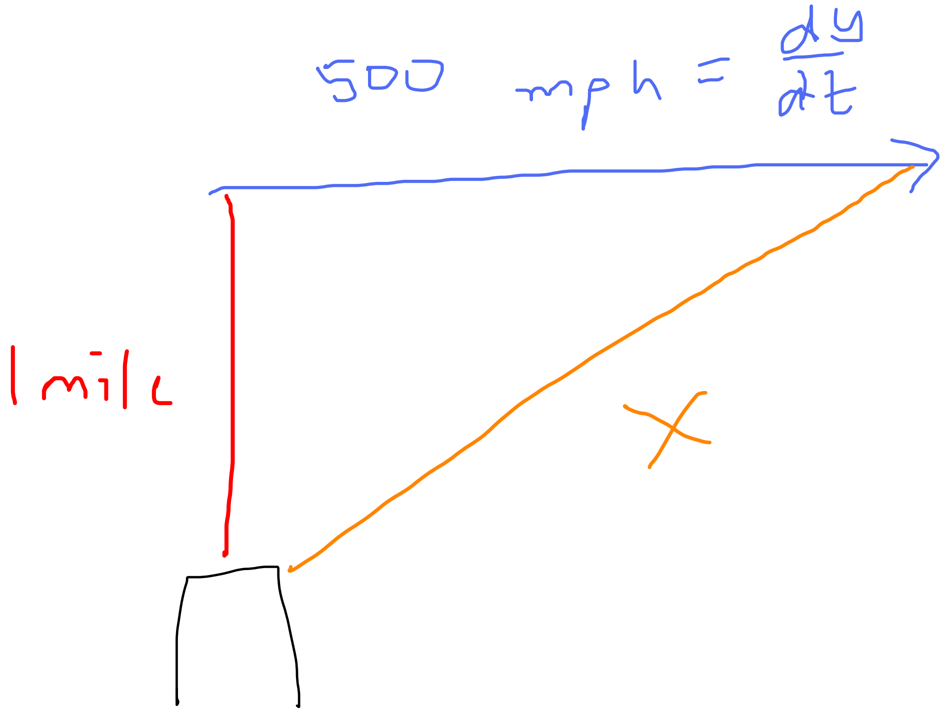
Disk method: For functions of x about the x axis

Shell method: For functions of x about the y axis.

Note: Integration makes related rates easy.

Example 4: A plane flying horizontally at an altitude of 1 mile and a speed of 500 m.p.h. passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the station.

Picture



Since $\frac{dy}{dt} = 500$, by integrating

with respect to t,

$$y = 500t + C.$$

If we suppose $x(0)=2$,

then since $y^2+1=x^2$,

$y(0)=\sqrt{3}$.

Then $C=\sqrt{3}$, so

$$x^2 = 1+(500t+\sqrt{3})^2$$

$$x = \sqrt{1+(500t+\sqrt{3})^2}$$

$$x'(t) = \frac{1}{2}(2)(500) \cdot \frac{500t+\sqrt{3}}{\sqrt{1+(500t+\sqrt{3})^2}}$$

We want $x'(t)$ when $x=2$, and this is $t=0$, so

$$x'(0) = 500 \cdot \frac{\sqrt{3}}{2} = \boxed{250\sqrt{3}}$$

You can check that this is the same answer you'd get using implicit differentiation